Finite-dimensional irreducible modules for an even subalgebra of $U_q(\mathfrak{sl}_2)$

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The algebra $U_q(\mathfrak{sl}_2)$

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The quantum algebra $U_q(\mathfrak{sl}_2)$ is the associative $\mathbb{F}$-algebra defined by generators e, f, k, k^{-1} and relations

$$kk^{-1} = k^{-1}k = 1,$$

$$kek^{-1} = q^2 e, \qquad kfk^{-1} = q^{-2} f,$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

This presentation is called the **Chevalley presentation** for $U_q(\mathfrak{sl}_2)$.

In 2006, Ito, Terwilliger, and Weng showed that $U_q(\mathfrak{sl}_2)$ has a presentation in generators $x, y^{\pm 1}, z$ and relations

$$yy^{-1} = y^{-1}y = 1,$$
$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$
$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1$$
$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

This presentation is called the **equitable presentation** for $U_q(\mathfrak{sl}_2)$.

Connections with $U_q(\mathfrak{sl}_2)$

The equitable presentation for $U_q(\mathfrak{sl}_2)$ has been used to show connections between $U_q(\mathfrak{sl}_2)$ and:

- Q-polynomial distance-regular graphs (Worawannotai, 2012),
- Leonard pairs (Alnajjar, 2011),
- Tridiagonal pairs (Ito/Terwilliger, 2007),
- the q-Tetrahedron algebra (Ito/Terwilliger, 2007; Funk-Neubauer, 2009; Miki, 2010),
- the universal Askey-Wilson algebra (Terwilliger, 2011),
- Poisson algebras (Jordan, 2010).

Bases for $U_q(\mathfrak{sl}_2)$

In the Chevalley presentation, the $\mathbb{F}$-vector space $U_q(\mathfrak{sl}_2)$ has a basis:

$$f^r k^s e^t \qquad r, t \in \mathbb{N}, \quad s \in \mathbb{Z}.$$

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Lemma (Terwilliger, 2011)

In the equitable presentation, the following is a basis for the $\mathbb{F}$-vector space $U_q(\mathfrak{sl}_2)$:

$$x^r y^s z^t \qquad r, t \in \mathbb{N}, \quad s \in \mathbb{Z}.$$

The algebra $\mathcal{A}$

Define $\mathcal{A}$ to be the $\mathbb{F}$-subspace of $U_q(\mathfrak{sl}_2)$ spanned by

$$x^r y^s z^t \qquad r, s, t \in \mathbb{N}, \quad r + s + t \text{ even.}$$

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Lemma (Bockting-Conrad and Terwilliger, 2013)

$\mathcal{A}$ is a subalgebra of $U_q(\mathfrak{sl}_2)$.

The elements ν_x, ν_y, ν_z

The relations from the equitable presentation for $U_q(\mathfrak{sl}_2)$ can be reformulated as:

$$q(1 - xy) = q^{-1}(1 - yx),$$
$$q(1 - yz) = q^{-1}(1 - zy),$$
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$$q(1 - zx) = q^{-1}(1 - xz).$$

We denote these elements ν_x, ν_y, ν_z respectively.

Observe that $\nu_x, \nu_y, \nu_z \in \mathcal{A}$.

Elements of $U_q(\mathfrak{sl}_2)$

Observe that $x^2, y^2, z^2 \in \mathcal{A}$. The elements ν_x, ν_y, ν_z and x^2, y^2, z^2 interact in the following way:

$$x^2\nu_y = q^4\nu_y x^2, \qquad x^2\nu_z = q^{-4}\nu_z x^2$$
$$y^2\nu_z = q^4\nu_z y^2, \qquad y^2\nu_x = q^{-4}\nu_x y^2$$
$$z^2\nu_x = q^4\nu_x z^2, \qquad z^2\nu_y = q^{-4}\nu_y z^2$$

$$x^2\nu_x - \nu_x x^2 = (q^2 - q^{-2})(\nu_y - \nu_z)$$
$$y^2\nu_y - \nu_y y^2 = (q^2 - q^{-2})(\nu_z - \nu_x)$$
$$z^2\nu_z - \nu_z z^2 = (q^2 - q^{-2})(\nu_x - \nu_y)$$

Generators for $\mathcal{A}$

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The $\mathbb{F}$-algebra $\mathcal{A}$ is generated by ν_x, ν_y, ν_z.

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In the same paper, Bockting-Conrad and Terwilliger posed the problem of finding a presentation for $\mathcal{A}$ in generators ν_x, ν_y, ν_z.

Relations involving ν_x, ν_y, ν_z

Proposition

In $U_q(\mathfrak{sl}_2)$, the elements ν_x, ν_y, ν_z satisfy

$$q^3\nu_x^2\nu_y - (q+q^{-1})\nu_x\nu_y\nu_x + q^{-3}\nu_y\nu_x^2 = (q^2-q^{-2})(q-q^{-1})\nu_x, \qquad (1)$$
$$q^3\nu_y^2\nu_z - (q+q^{-1})\nu_y\nu_z\nu_y + q^{-3}\nu_z\nu_y^2 = (q^2-q^{-2})(q-q^{-1})\nu_y, \qquad (2)$$
$$q^3\nu_z^2\nu_x - (q+q^{-1})\nu_z\nu_x\nu_z + q^{-3}\nu_x\nu_z^2 = (q^2-q^{-2})(q-q^{-1})\nu_z, \qquad (3)$$

and

$$q^{-3}\nu_y^2\nu_x - (q+q^{-1})\nu_y\nu_x\nu_y + q^3\nu_x\nu_y^2 = (q^2-q^{-2})(q-q^{-1})\nu_y, \qquad (4)$$
$$q^{-3}\nu_z^2\nu_y - (q+q^{-1})\nu_z\nu_y\nu_z + q^3\nu_y\nu_z^2 = (q^2-q^{-2})(q-q^{-1})\nu_z, \qquad (5)$$
$$q^{-3}\nu_x^2\nu_z - (q+q^{-1})\nu_x\nu_z\nu_x + q^3\nu_z\nu_x^2 = (q^2-q^{-2})(q-q^{-1})\nu_x. \qquad (6)$$

Relations involving ν_x, ν_y, ν_z

Proposition

In $U_q(\mathfrak{sl}_2)$, the elements ν_x, ν_y, ν_z satisfy

$$\nu_x \frac{q\nu_y\nu_z - q^{-1}\nu_z\nu_y}{q - q^{-1}} = \nu_x - q^{-2}\nu_y - q^2\nu_z + \frac{q^2\nu_y\nu_z - q^{-2}\nu_z\nu_y}{q - q^{-1}}, \qquad (7)$$

$$\frac{q\nu_y\nu_z - q^{-1}\nu_z\nu_y}{q - q^{-1}}\nu_x = \nu_x - q^2\nu_y - q^{-2}\nu_z + \frac{q^2\nu_y\nu_z - q^{-2}\nu_z\nu_y}{q - q^{-1}}, \qquad (8)$$

and the relations (9)–(12) *obtained from these by cyclically permuting* $\nu_x \to \nu_y \to \nu_z \to \nu_x$.

A presentation for $\mathcal{A}$

Theorem (Lynch, 2014+)

The $\mathbb{F}$-algebra $\mathcal{A}$ is isomorphic to the $\mathbb{F}$-algebra defined by generators ν_x, ν_y, ν_z and the relations (1)–(12) from the previous two slides.

Representation theory of $U_q(\mathfrak{sl}_2)$

We now turn our attention to the representation theory of $\mathcal{A}$.

First, we recall some facts about the representation theory of $U_q(\mathfrak{sl}_2)$.

Finite dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules

For $n \in \mathbb{N}, \ \varepsilon \in \{1, -1\}$, there exists an irreducible $U_q(\mathfrak{sl}_2)$-module $L(n, \varepsilon)$ of dimension n which has a basis $\{v_i\}_{i=0}^{n}$ such that

$$\begin{aligned} \varepsilon x.v_i &= q^{2i-n} v_i + (q^n - q^{2i-2-n}) v_{i-1}, \\ \varepsilon y.v_i &= q^{n-2i} v_i, \\ \varepsilon z.v_i &= q^{2i-n} v_i + (q^{-n} - q^{2i+2-n}) v_{i+1}, \end{aligned}$$

where $v_{-1} = v_{n+1} = 0$.

Moreover, every finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module is isomorphic to some $L(n, \varepsilon)$.

Induced modules of $\mathcal{A}$

For each $n \in \mathbb{N}$, $\varepsilon \in \{1, -1\}$, observe that $L(n, \varepsilon)$ has an induced $\mathcal{A}$-module structure.

For $n \in \mathbb{N}$, the $\mathcal{A}$-modules $L(n, 1)$ and $L(n, -1)$ are isomorphic.

We denote by $L(n)$ the common $\mathcal{A}$-module structure of $L(n, 1)$ and $L(n, -1)$.

Facts about $L(n)$

We note a few facts about the $\mathcal{A}$-module $L(n)$:

- $L(n)$ is irreducible as an $\mathcal{A}$-module.
- The actions of ν_x, ν_y, ν_z on $L(n)$ are nilpotent.
- The actions of x^2, y^2, z^2 on $L(n)$ are diagonalizable.

Facts about finite-dimensional irreducible $\mathcal{A}$-modules

What about arbitrary finite-dimensional irreducible $\mathcal{A}$-modules?

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Lemma

Let V be a finite-dimensional irreducible $\mathcal{A}$-module. Then the actions of ν_x, ν_y, ν_z on V are nilpotent.

Facts about finite-dimensional irreducible $\mathcal{A}$-modules

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Let V be a finite-dimensional irreducible $\mathcal{A}$-module. Then the actions of x^2, y^2, z^2 on V are diagonalizable.

Finite-dimensional irreducible $\mathcal{A}$-modules

Theorem (Lynch, 2014+)

Let V be a finite-dimensional irreducible $\mathcal{A}$-module. Then V is isomorphic to $L(n)$ for some $n \in \mathbb{N}$.

Future work

- Find a presentation for $\mathcal{A}$ in generators x^2, y^2, z^2.
- Classify the center of $\mathcal{A}$.
- Investiate the induced $\mathcal{A}$-modules from the $U_q(\mathfrak{sl}_2)$-modules related to tridiagonal pairs, the q-tetrahedron algebra, etc.

Thank you!