Mathematicians' Example-Related Activity in Formulating Conjectures

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Introduction and Motivation

- There is substantial evidence that students at all levels struggle with learning to proof (Healy & Hoyles, 2000; Kloosterman & Lester, 2004; Knuth, Choppin, & Bieda, 2009; Porteous, 1990)
- One way to gain insight into how better to help students is to study the work of mathematicians (Carlson & Bloom, 2005; Weber, 2008; Lockwood, et al. 2012)

Introduction and Motivation

- Researchers have reported that students and mathematicians display strategic uses of examples which benefit their proof-related activities (Ellis, et al., 2012; Garuti, Boreo & Lemut, 1998; Pedemonte, 2007; Sandefur, et al., 2013; Weber, 2008)
- Our work builds on such studies by seeking to identify fruitful aspects of example-related activity in the development of conjectures

Introduction and Motivation

- "Most mathematicians spend a lot of time thinking about and analyzing particular examples…It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" Epstein & Levy (1995)
- We broadly define an example as Bills and Watson (2008) do, as "any mathematics object from which it is expected to generalize" (p.78)

Research Question

- With what kinds of example-related activity do mathematicians engage as they develop conjectures?

Interesting Numbers Task

- Most positive integers can be expressed as a sum of two or more consecutive positive integers.
  - For example, 24 = 7 + 8 + 9 and 51 = 25 + 26.

  A positive integer that cannot be expressed as the sum of two or more consecutive positive integers is, therefore, interesting.

- What are all the interesting numbers?

Methods - Participants

- 13 mathematicians participated in hour-long interviews
  - 7 professors, 3 postdocs, 3 lecturers
  - 12 hold a Ph.D. in mathematics, 1 holds a Ph.D. in computer science
  - Researchers in areas including topology, number theory, and analysis
- Each mathematician worked on the Interesting Numbers task, and one other task as time allowed

Methods – Tasks

- The Interesting Numbers task was chosen to be
  - Accessible to the mathematicians, but not trivial
  - Accessible to the interviewer
  - Open-ended to facilitate conjecturing

Methods - Analysis

- Mathematicians' written work and audio were recorded using Livescribe pens
- We coded each interview for example types, uses, and strategies using the framework of Lockwood, et al (2012)
- Members of the research team discussed emerging themes and additional types, uses, and strategies to add to the framework

Task Solution

- Correct conjecture:
  - The interesting numbers are the powers of 2
- We give a proof of the contrapositive:
  - The non-interesting numbers are the non-powers of 2

Proof

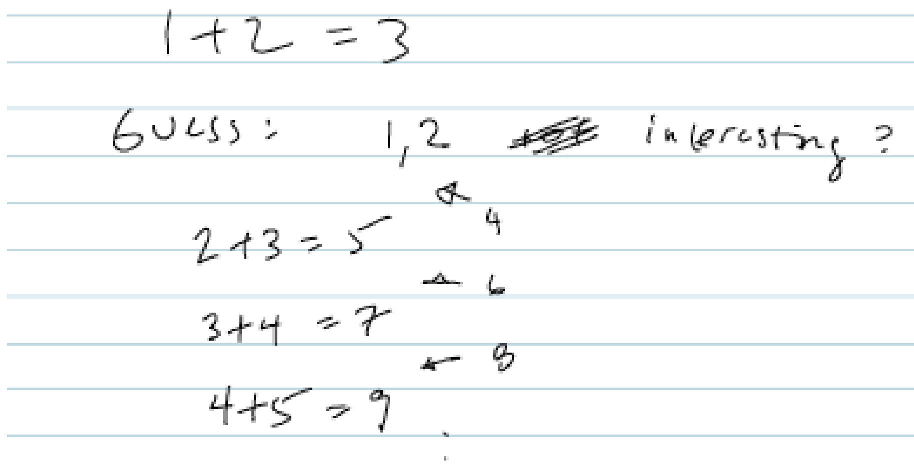
- Not interesting → Not a power of 2
  - The sum of two or more consecutive positive integers has an odd factor > 1
- Not a power of 2 → Not interesting
  - If *N* has an odd factor k > 1, it can be written as the sum of $\min\{k, 2N/k\}$ consecutive positive integers
- Therefore, the interesting numbers are the powers of 2

Results

- 10 of the 13 mathematicians engaged in “Data Collection” activity
  - Systematically and exhaustively going through every example in a finite sequence to gather information
  - Reflecting back on their organized example list to formulate a conjecture

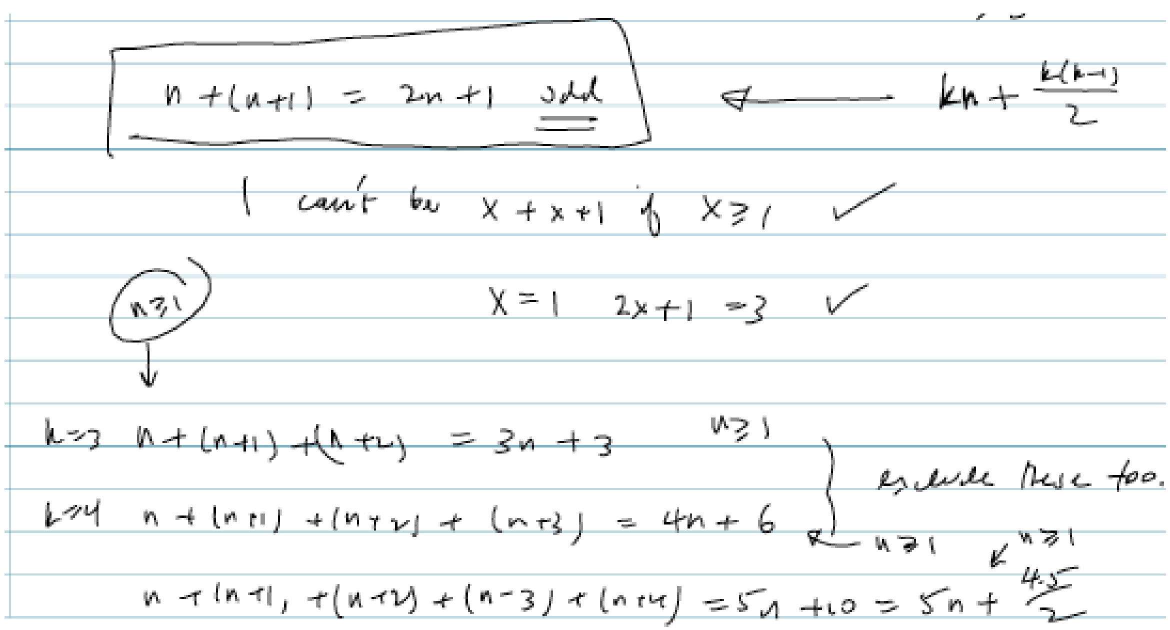
Dr. Sullivan's Data Collection

- After trying the first few 2-number sums, Dr. Sullivan recognized that odd numbers greater than 1 could not be interesting



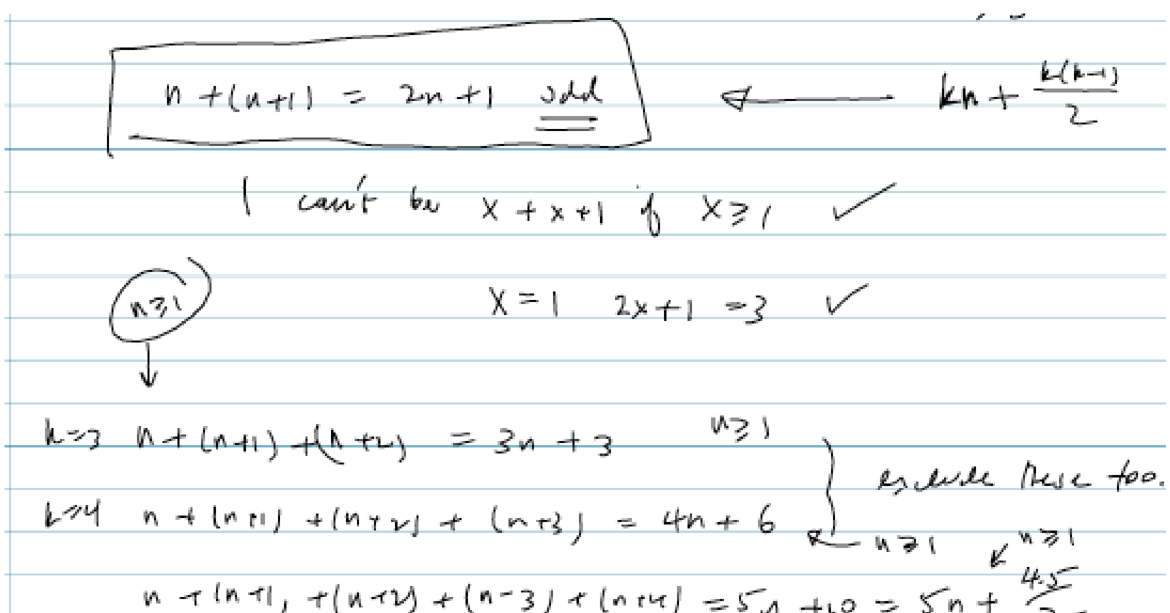
Dr. Sullivan's Data Collection

- Dr. Sullivan generalized this idea algebraically, looking at sums of 3, 4, and 5 consecutive numbers



Dr. Sullivan’s Data Collection

- Each case gave him an algebraic expression representing numbers that were not interesting

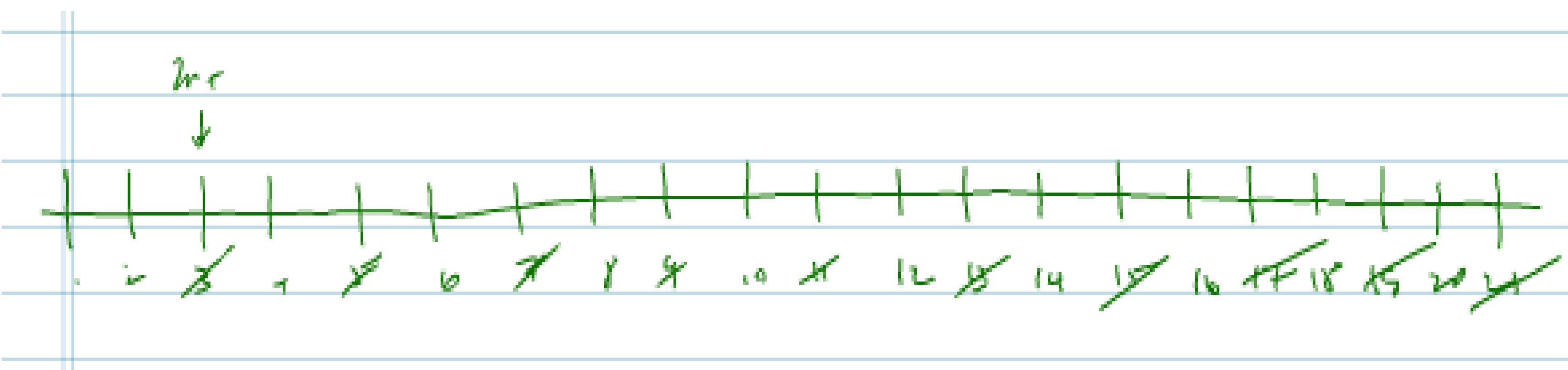


Dr. Sullivan's Data Collection

- However, Dr. Sullivan's algebraic manipulation did not illuminate a general conjecture for him
- *Dr. Sullivan*: Okay. So at this point, I would start over and try to do something a little more visual.

Dr. Sullivan's Data Collection

- On his number line, he systematically crossed out the numbers of the form *2n+1, 3n+3, 4n+6, 5n+10*



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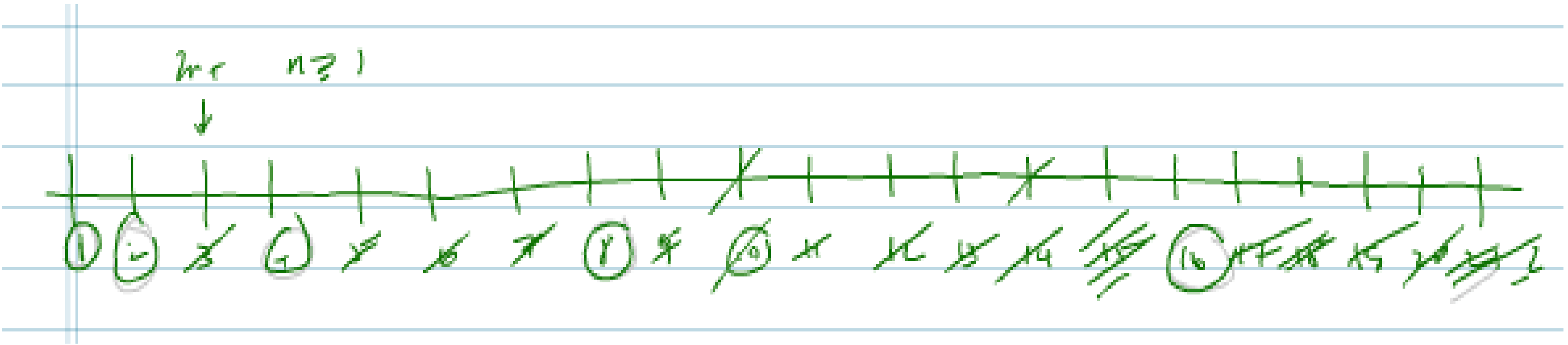
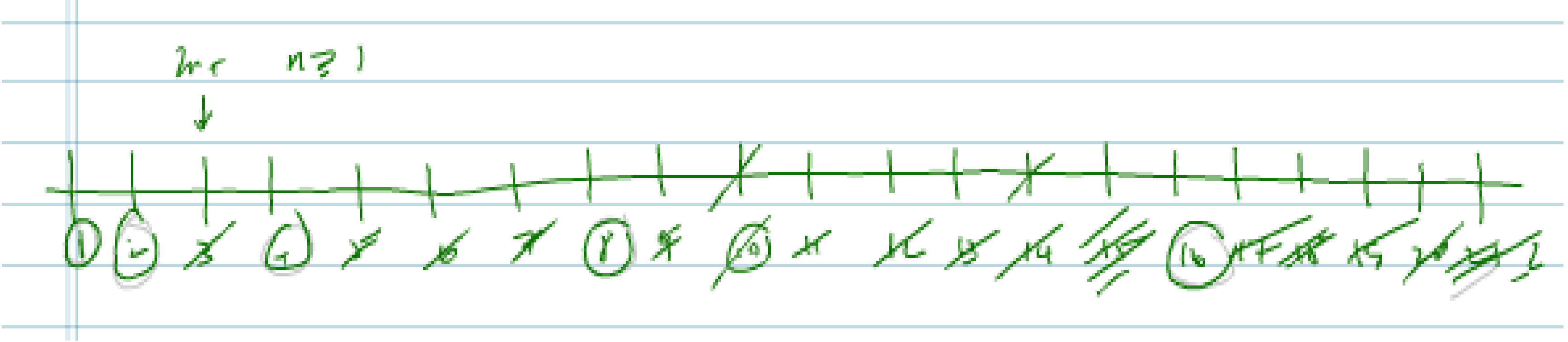


Dr. Sullivan's Data Collection

- *Dr. Sullivan*: Well, the answer does kind of pop out that it's the powers of 2, doesn't it?



Dr. Sullivan's Data Collection

- Dr. Sullivan's work suggests that Data Collection helped facilitate the efficient formulation of the conjecture

Dr. Weisman's Data Collection

- As another, more extreme, instance of data collection, we consider the work of Dr. Weisman
- Dr. Weisman found that the non-interesting numbers were of the form $(n-m)(n+m-1)/2$ with $n > m > 0$
- He proceeded to make a table of the first few numbers of that form

n \ m	1	2	3	4	
1	/				
2	$\frac{1\cdot 2}{2}$ ~~1~~	/			
3	$\frac{2\cdot 3}{2}$ 3	$\frac{1\cdot 4}{2}$ ~~2~~	/		
4	$\frac{3\cdot 4}{2}$ 6	$\frac{2\cdot 5}{2}$ 5	$\frac{1\cdot 6}{2}$ ~~3~~	/	
5	$\frac{4\cdot 5}{2}$ 10	$\frac{3\cdot 6}{2}$ 9	$\frac{2\cdot 7}{2}$ 7	$\frac{1\cdot 8}{2}$ ~~4~~	
6	$\frac{5\cdot 6}{2}$ 15	$\frac{4\cdot 7}{2}$ 14	$\frac{3\cdot 8}{2}$ 12	$\frac{2\cdot 9}{2}$ 9	/
7	$\frac{[illegible]\cdot 1}{2}$ 21	$\frac{[illegible]\cdot 8}{2}$ 20	$\frac{[illegible]\cdot 9}{2}$ 18	$\frac{3\cdot 10}{2}$ 15	$\frac{2\cdot 11}{2}$ 11

Dr. Weisman's Data Collection

- By examining his table, Dr. Weisman deduced some patterns in the non-interesting numbers, but found nothing conclusive
- *Dr. Weisman*: Okay, well, I'm a believer in generating some data...what I'm gonna do is make an even bigger version of this table. And just, just look to see what numbers show up.

	1	2	3	4	5	6	7	8	9	10	11
1	/										
2	/	/									
3	3	/	/								
4	6	5	/	/							
5	10	9	7	/	/						
6	15	(14)	(12)	9	/	/					
7	21	(20)	(18)	15	11	/	/				
8	(28)	27	25	(22)	18	13	/	/			
9	(36)	35	33	30	(26)	21	15	/	/		
10	45	(44)	42	39	35	(30)	(24)	17	/	/	
11	55	54	52	49	45	(40)	(34)	27	19	/	/
12	66	65	63	60	56	51	45	(38)	30	21	
13	78	77	75	72	68	63	57	(50)	(42)	33	
14	71	90	88	85	81	76	70	63	55	(46)	
15	105	104	102	99	95	90	84	77	69	60	
										70	

Dr. Weisman's Data Collection

- By referencing this larger table, Dr. Weisman made a conclusive list of the non-interesting numbers up to 50
- From that list, he noticed the pattern in the missing numbers, which led him to the correct conjecture

Other Effects of Data Collection

- In addition to helping mathematicians formulate a correct conjecture, we mention two other ways in which Data Collection supported mathematicians' conjecturing:
  - Lemma Development
  - Preliminary Conjecture Breaking

Lemma Development

- For some of the mathematicians, observations from generated data lead to lemmas, which in turn informed the development of conjectures
- We illustrate this process using the work of Dr. Taylor

Lemma Development

- Dr. Taylor first looked at the numbers 1-14, trying to write each one as a sum of consecutive numbers
- He noticed that
  - Odd numbers were sums of 2 consecutive
  - Multiples of 6 were sums of 3 consecutive
  - Numbers congruent to 2 mod 4 (other than 2) were sums of 4 consecutive

Lemma Development

- He then proved lemmas stating that odd numbers, multiples of 6, and numbers congruent to 2 mod 4 were not interesting
- These lemmas allowed Dr. Taylor to restrict his attention to multiples of 4, which led to the development of the full conjecture

Preliminary Conjecture Breaking

- In other cases, data collection allowed the mathematicians to find examples which broke preliminary conjectures, which led to the articulation of more accurate conjectures
- We illustrate this process using the work of Dr. Hughes

Preliminary Conjecture Breaking

- After looking at the numbers 1-6, Dr. Hughes conjectured that the interesting numbers were the non-primes

1 and 2 does not count.

3 = 1+2

4 = is not

5 = 2+3

6 = is not

7 = 3+4

8 = is not

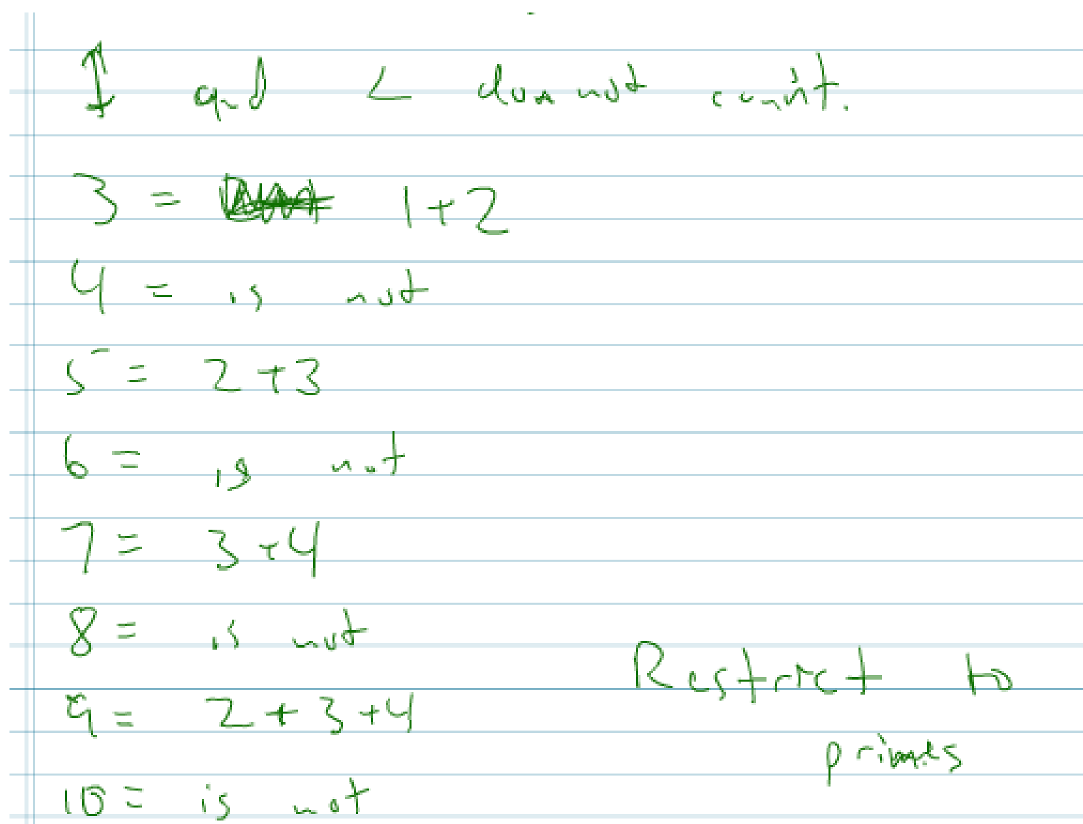
9 = 2+3+4

10 = is not

Restrict to primes

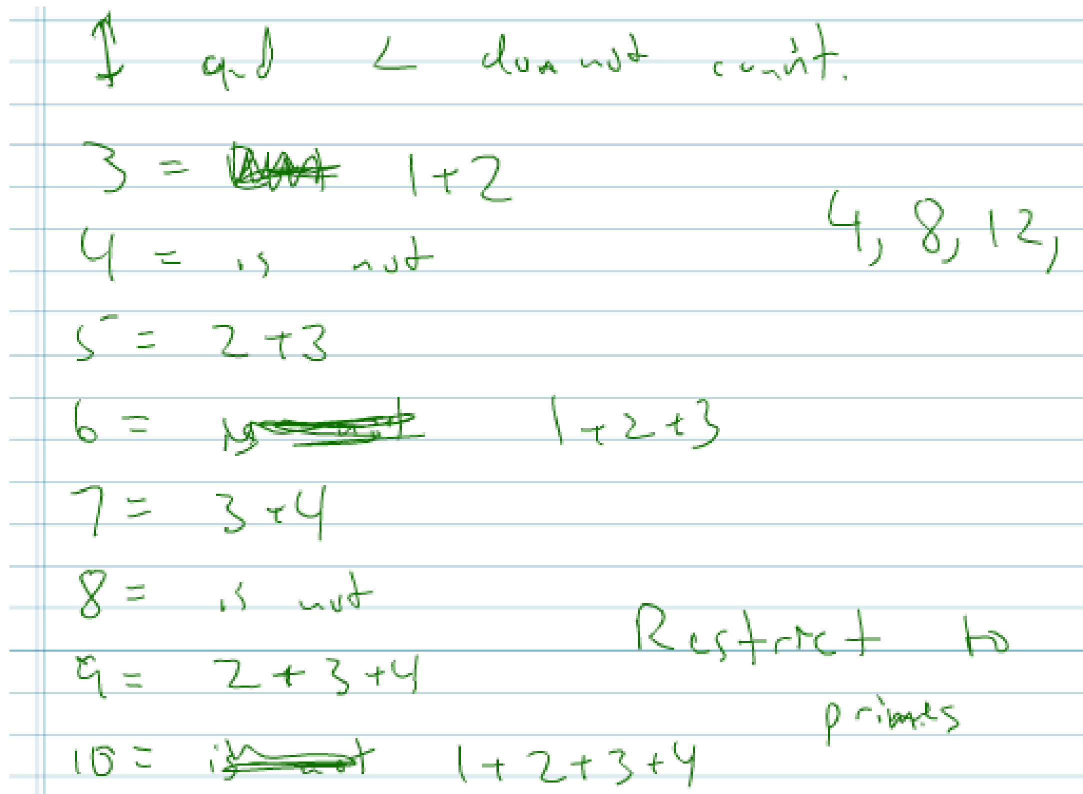
Preliminary Conjecture Breaking

- He continued on to look at the numbers 7-10 before realizing that he was mistaken about 6



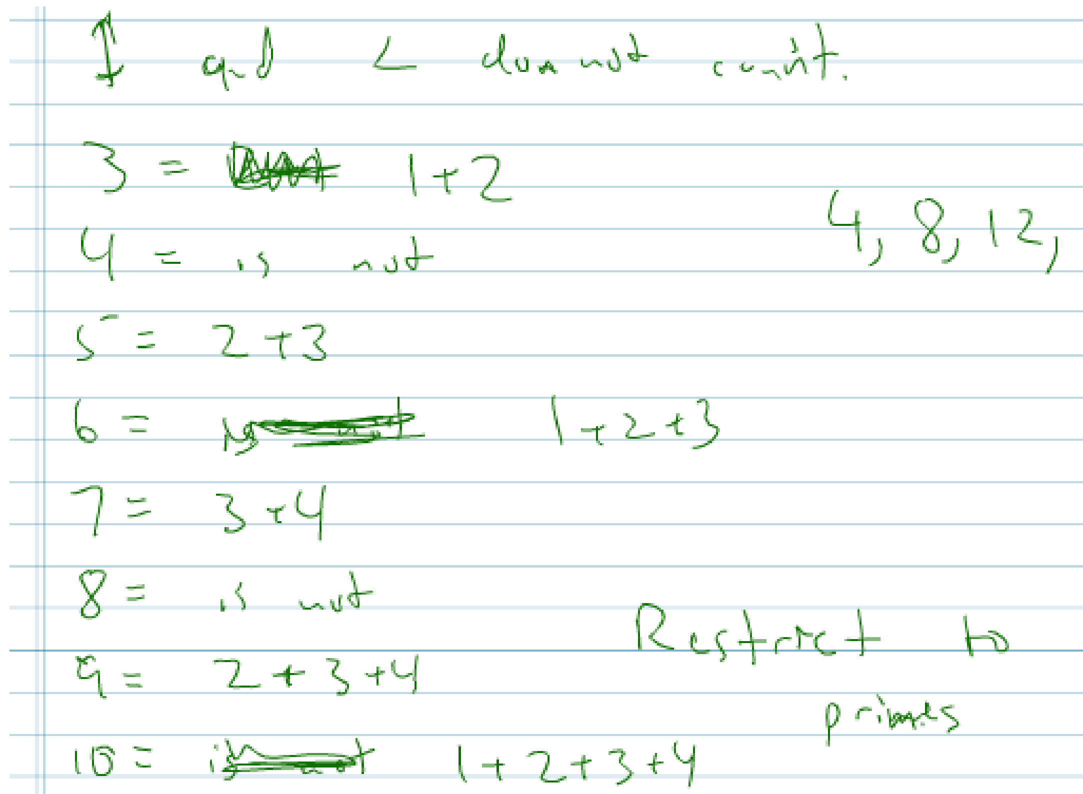
Preliminary Conjecture Breaking

- *Dr. Hughes*: Oh, 1, 2,… 1 plus 2 plus 3. Right. Revise conjecture. So far, the interesting numbers are 4, 8, …. It looks like it's the multiples of 4.



Preliminary Conjecture Breaking

- After looking at 11, 12, and 13 and discovering that 12 was not interesting, Dr. Hughes revised his conjecture once more (to a correct conjecture)



Other approaches

- The 3 mathematicians who did not use a Data Collection approach used few, if any, examples
- They each used a primarily algebraic approach
- Though this approach yielded a correct conjecture for 2 of the 3 mathematicians, it took them longer
  - 38 minutes on average for algebra users
  - 16 minutes on average for data collectors

Implications for Students

- For some of the mathematicians, the careful generation of examples and subsequent reflection on them enabled them to formulate conjectures effectively and efficiently
  – Students may benefit from generating comprehensive sets of data that they can survey in search of patterns

Implications for Students

- These mathematicians engaged in deliberate and strategic example generation, which stands in contrast to less systematic behavior often found in students' work with examples
  - For students, there may be value in helping them learn to be more strategic and methodical in their use of examples

Implications for Students

- In some of the mathematicians, we saw an immediate application of algebraic techniques that were less efficacious for conjecturing than the Data Collection was
  - These findings suggest that students should be encouraged to engage with finding concrete examples when conjecturing and not simply to apply algebraic formulas and techniques.

Thank You!

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http://examples.wceruw.org